A HOMOLOGICAL APPROACH FOR COMPUTING THE TANGENT SPACE OF THE DEFORMATION FUNCTOR OF CURVES WITH AUTOMORPHISMS.

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ABSTRACT. We give an alternative approach to the computation of the dimension of the tangent space of the deformation space of curves with automorphisms. A homological version of the local-global principle similar to the one of J.Bertin, A. Mézard is proved, and a computation in the case of ordinary curves is obtained, by application of the results of S. Nakajima for the Galois module structure of the space of 2-holomorphic differentials on them.

1. INTRODUCTION

Let X be a non-singular curve of genus $g \geq 2$ defined over an algebraic closed field of positive characteristic, together with a subgroup G of the automoprhism group. In [1] J.Bertin, A. Mézard proved that the equivariant cohomology of Grothendieck $H^1(G, \mathcal{T}_X)$ measures the tangent space of the global deformation functor of curves with automorphisms. This dimension is a measure in how many directions a curve can be deformed together with a subgroup of an automoprhism group.

Since the genus g of X is $g \geq 2$ the edge homomorphisms of the spectral sequence of Grothendieck [4, 5.2.7] give as that

$$H^1(G, \mathcal{T}_X) = H^1(X, \mathcal{T}_X)^G,$$

i.e. the first equivariant cohomology equals the G-invariant space of the "tangent space" of the moduli space at X. J. Bertin and A. Mézard were successful to compute $H^1(G, \mathcal{T}_X)$, using a version of equivariant Chech theory, and prove a local-global theorem.

It is tempting, in order to compute the G-invariants of the space $H^1(G, \mathcal{T}_X)$ to use Serre's duality to pass to the space $H^0(G, \Omega_X^{\otimes 2})$. One should be careful using this approach, because since we are considering the dual space of $H^1(X, \mathcal{T}_X)$ it is not the functor of invariants that we have to consider, but the the adjoint functor, *i.e.*, the functor of covariants.

This article consists of two parts. In first part, we use the normal basis theorem for Galois extensions and the explicit form of Serre duality in terms of repartitions [5, 7.14.2],[11, I.5], in order to compute the covariant elements of $H^0(X, \Omega_X^{\otimes 2})$. This leads us to a homological version of the local-global theorem of J. Bertin A. Mézard (4). The duality between the homological and the cohomological approach is emphasized by defining a cap product

$$H^p(G, H^1(G, \mathcal{T}_X)) \times H_q(G, H^0(X, \Omega_X^{\otimes 2})) \to H_{q-p}(G, k).$$

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In second part we try to apply known results on the Galois module structure of the space of holomorphic differentials, in order to compute covariant elements. We have to notice here, that, as far the author knows, if the characteristic p divides $|G|$, the Galois module structure of $H^0(X, \Omega^{\otimes s})$ is far from being understood, and there are only partial results mainly in the case of tame ramification [6],[9],[7], or in the case of ordinary curves.

More precisely, S. Nakajima in [8] studied the Galois module structure of the "semi-simple part" of $H^0(X, \Omega(-D)$ with respect to the Cartier operator if G is a p-group and D is an effective G-invariant divisor on X. Thus, for the Zariski dense set in the moduli space of curves of genus g, of ordinary curves (curves the Galois module structure of $H^0(X, \Omega^{\otimes 2})$ is known.

The tangent space to the deformation functor of ordinary curves was studied by G. Cornelissen and F. Kato in [3]. We obtain a weaker result than their result by using the methods of S. Nakajima.

2. Computations

Denote by K_X the function field of the curve X. Let $\mathcal{K}_X$ be the constant sheaf K_X, and consider the exact sequence of sheaves

$$0 \to \emptyset_X \to \mathcal{K}_X \to \frac{\mathcal{K}_X}{\emptyset_X} \to 0. \tag{1}$$

The sheaf $\frac{\mathcal{K}_X}{\emptyset_X}$ can be expressed in the form

$$\frac{\mathcal{K}_X}{\emptyset_X} = \bigoplus_{P \in X} i_*(K_X/\emptyset_P),$$

where $i : \mathrm{Spec}\emptyset_P \to X$ is the inclusion map.

We tensor the sequence (1) with the sheaf $\Omega_X^{\otimes 2}$ over $\emptyset_X$ and get the sequence:

$$0 \to \Omega_X^{\otimes 2} \to \mathcal{K}_X \otimes_{\emptyset_X} \Omega_X^{\otimes 2} \to \bigoplus_{P \in X} i_*(K_X/\emptyset_P) \otimes \Omega_X^{\otimes 2} \to 0.$$

We will denote by $\mathcal{M}^{\otimes 2} = \mathcal{K}_X \otimes_{\emptyset_X} \Omega_X^{\otimes 2}$ the sheaf of meromorphic 2-differentials and by $\Omega_P^{\otimes 2} = \Omega_X^{\otimes 2} \otimes_{\emptyset_X} \emptyset_P$. Thus we might write

$$\bigoplus_{P \in X} i_*(K_X/\emptyset_P) \otimes \Omega_X^{\otimes 2} = \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}).$$

We apply the global section functor:

$$0 \to \Gamma(X, \Omega_X^{\otimes 2}) \to \Gamma(X, \mathcal{K}_X \otimes_{\emptyset_X} \Omega_X^{\otimes 2}) \to \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}) \to H^1(X, \Omega_X^{\otimes 2}) \to \cdots$$

Since X is a curve of genus $g \geq 2$ we have that $H^1(X, \Omega_X^{\otimes 2}) = 0$ and if we denote by $\Omega = \Gamma(X, \Omega_X^{\otimes 2})$ and $M = \Gamma(X, \mathcal{M}^{\otimes 2})$ the spaces of global sections of homomorphic and meromorphic differentials we have:

$$0 \to \Omega \to M \to \Gamma\left(X, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})\right) \to 0. \tag{2}$$

Lemma 2.1. *The G-module M as a $K_Y[G]$ module is projective.*

Proof. Let w be a meromorphic differential of the curve $Y = X/G$, and denote by K_Y the function field of the curve Y. The lift $\pi_* w$ is a G-invariant meromorphic differential on X, and M can be recovered as the set of the expressions

$$M = \{f \cdot \pi_*(w), \quad f \in K_X\}.$$

We want to apply the functor of covariants, *i.e.*, to tensor with $K_Y \otimes_{K_Y[G]}$. We notice first that by the normal basis theorem [12, 6.3.7 p.173] for the Galois extension K_X/K_Y we obtain that $K_X \cong K_Y[G]$ as a Galois module, thus M is isomorphic to $K_Y[G]$ as a $K_Y[G]$-module and the desired result follows. $\square$

We consider the long exact homology sequence arising from (2) after taking the functor of covariants:

$$(3) \qquad 4 \cdots \to H_1(G, M) \to H_1(G, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})) \to \Omega_G \xrightarrow{\alpha} M_G \to$$

$$\to \Gamma(X, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}))_G \to 0.$$

Since $M \cong K_Y[G]$ we have $H_1(G, M) = 0$ and $M_G = \{f \cdot \pi_*(w)\}$, with $f \in K_Y$.

Thus

$$\Omega_G = H_1(G, \Gamma(X, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}))) \oplus \mathrm{Im}\alpha.$$

Remark: If the order $|G|$ of the group G is prime to the characteristic p then the order $|G|$ is invertible in the module $\Gamma(X, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}))$ and the first homology is zero, therefore

$$\Omega_G = \mathrm{Im}\alpha.$$

Proposition 2.2. *Let $b_1, \ldots, b_r$ be the set of ramification points of the cover $X \to Y$, and let $G_i = G(b_i)$ be the corresponding decomposition groups. The following holds:*

$$H_1(G, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})) = \bigoplus_{i=1}^{r} H_1(G_i, \mathcal{M}^{\otimes 2}/\Omega_{b_i}^{\otimes 2})).$$

Proof. Let P be a point of X, and let t_P be a local uniformizer at the point P. Consider an element $a = \sum_{P \in X} a_P P \in \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}))$. The polar part of a at P is equal to $\sum_{\nu=-n}^{-1} \frac{a_\nu}{t^\nu}$. For an element $g \in G$ we have that

$$g\left(\sum_{\nu=-n}^{-1} \frac{a_\nu}{t^\nu}\right) = \sum_{\nu=-n}^{-1} \frac{a_\nu}{g(t)^\nu}.$$

The element $g(t)$ is the local uniformizer at the point $g(P)$. This proves that the action of the element $g \in G$ on a is of the form

$$g(\sum_{P \in X} a_P P) = \sum_{P \in X} a_P g(P).$$

Let $M_P = i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})$ be the summand corresponding to the point P, and let $G(P) = \{g \in G : g(P) = P\}$ be the decomposition group at the point P. We consider the induced module, seen as a subspace of $\oplus_{P \in X} \mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}$,

$$\mathrm{Ind}_{G(P)}^{G} M_P = K_Y[G] \otimes_{K_Y[G(P)]} M_P = \bigoplus_{g \in G/G(P)} M_{g(P)}.$$

Shapiro's lemma [12, 6.3.2] implies that

$$H_1(G, \mathrm{Ind}_{G(P)}^{G} M_P) = H_1(G(P), M_P).$$

Thus if P is not a ramification point it does not contribute to the cohomology, and the desired formula comes by the sum of the contributions of the ramification groups.

□

We have proved that the following sequence is exact:

$$0 \to \bigoplus_{i=1}^{r} H_1(G(b_i), \mathcal{M}^{\otimes 2}/\Omega_{b_i}^{\otimes 2})) \to \Omega_G \to \mathrm{Im}\alpha \to 0 \tag{4}$$

which is exactly the dual sequence of J. Bertin, A. Mézard [1, p . 206].

Proposition 2.3. *Let $b_1, \ldots, b_r$ be the ramification points of the cover $\pi : X \to Y$, and assume that the groups in the ramification filtration at each ramification point b_k have orders*

$$e_0^{(k)} \geq e_1^{(k)} \geq \cdots \geq e_{n_k}^{(k)} > 1.$$

The dimension of the space $\mathrm{Im}\alpha$ *is given by:*

$$\dim_k \mathrm{Im}\alpha = 3g_y - 3 + \sum_{k=1}^{r} \left\lceil 2 \sum_{i=1}^{n_k} \frac{e_i^{(k)} - 1}{e_0^{(k)}} \right\rceil.$$

Proof. We are looking for elements of the form $f\pi^*(w)$, $f \in K_Y$, such that $\mathrm{div}_X f\pi^*(w) \geq 0$. We know that if w is a 2-differential then:

$$\mathrm{div}(\pi^*(w)) = \pi^*(\mathrm{w}) + 2R,$$

where R is the ramification divisor of $\pi : X \to Y$. Therefore $\pi^*(w)$ is holomorphic if and only if $\pi^*(\mathrm{w} + 2R) \geq 0$.

We will push forward again and we will use Riemann-Roch on Y. We want to compute the dimension of the space

$$L(K + \pi_*(2R)/|G|),$$

where K is the canonical divisor.

The ramification divisor is

$$R = \sum_{k=1}^{r} \sum_{i=1}^{n_k} (e_i^{(k)} - 1) b_k,$$

and Riemann-Roch theorem implies that

$$\dim_k \mathrm{Im}\alpha = 3g_y - 3 + \sum_{k=1}^{r} \left\lceil 2 \sum_{i=1}^{n_k} \frac{e_i^{(k)} - 1}{e_0^{(k)}} \right\rceil.$$

□

Proposition 2.4. *For a ramification point b we have $H_1(G(b), M_b) \cong V(b)_G$, where $V(b)$ is a finite k-vector space, with known G-module structure.*

Proof. The space M_b consists of elements in $1/tk[1/t]$. Let $f(1/t) \in k[1/t]$ be a G-invariant element. Then we can consider the direct product of G-modules:

$$k((t)) = 1/tf(1/t)k(1/t) \bigoplus L(v(f)),$$

where $k((t))$ is the field of formal Laurent series, v is the valuation of the field $k((t))$, and $L\big(v(f)\big) = \{g \in k((t)), v(g) + v(f(1/t)) \geq 0\}$. Theorem 90 of Hilbert implies that $H_1\big(G(b), k((t))\big) = 0$, and since

$$H_1\big(G(b), k((t))\big) = H_1\big(G(b), f(1/t)k(1/t)\big) \bigoplus H_1\big(G(b), L(v(f))\big),$$

we have that $H_1\big(G(b), 1/tf(1/t)k(1/t)\big) = 0$. We now consider the short exact sequence:

$$0 \to 1/tk[1/t] \xrightarrow{\times f(1/t)} f(1/t)1/tk[1/t] \to V(b) \to 0,$$

and by a dimension shifting argument we obtain the desired result. $\square$

3. Definition of a cap product.

In this section we will define a cap product between the homology and cohomology groups of $A := H^1(X, \mathcal{T}_X)$ and $B := H^0(X, \Omega_X^{\otimes 2})$. Serre duality implies the existence of a trace function

$$A \times B \to k.$$

Thus, a cap product can be defined ([2, p.113])

$$H^p(G, A) \otimes H_q(G, B) \to H_{q-p}(G, A \otimes B) \xrightarrow{\mathrm{tr}} H_{q-p}(G, k).$$

In particular, the above cap product gives us the pairing

$$H^1(G, A) \otimes H_1(G, B) \to k$$

connecting the homological and cohomological approaches to the theory.

Remark: Since G is a finite group, the Tate cohomology groups are defined, and they seem a more natural tool for the study of cap products and duality. Unfortunately we are interested for the computation of invariants and co-invariants, *i.e.* for low index cohomology groups, and Tate cohomology can not be applied here.

4. Ordinary Curves

Let D be an effective divisor on the curve X, we will denote by

$$\Omega_X(-D) = \{f \in k(X) : \mathrm{div}(f) + \mathrm{div}(\omega) \geq -D\}.$$

S. Nakajima in [8] provided us with a method for computing the Galois module structure of the semisimple part of $\Omega_X(-D)$ with respect to the Cartier operator on X.

If the curve X is ordinary then the semisimple part of the Cartier operator is identified with the space $\Omega_X(-D)$ itself. We are interested in computing the space of covariants $\Omega_G^{\otimes 2}$ of the holomorphic 2-differentials.

The space of holomorphic 2-differentials can be identified with the space

$$\{f \in k(X) : \mathrm{div}(f) + 2\mathrm{div}(\omega) \geq 0\} = \{\mathrm{div}(f) + \mathrm{div}(\omega) \geq -\mathrm{div}(\omega)\}$$

Lemma 4.1. *There is a G-invariant differential ω in X, such that* $\mathrm{div}(\omega)$ *is effective, and has support that does not intersect the branch locus.*

Proof. Let $b_1, \ldots, b_r$ be the ramification points of the cover $\pi : X \to Y = X/G$.

Let ϕ_1 be an arbitrary meromorphic differential on Y. We will select a meromorphic differential $\phi = f\phi_1$ on the curve Y such that $\mathrm{div}(\phi) = \mathrm{div}(f\phi_1) + A \geq 0$, where A is the divisor

$$A := \sum_{i=1}^{r} \left\lceil \sum_{i=0}^{\infty} \frac{e_i(b_i) - 1}{e_0(b_i)} \right\rceil b_i.$$

Notice that if we assume that we are working on an ordinary curve then $e_2(b_i) = 0$ [10], and the above divisor can be written as

$$A = \sum_{i=1}^{r} \lambda_i b_i,$$

where $\lambda_i = 1$ if b_i is ramified tamely and $\lambda_i = 2$ if b_i is ramified wildly.

This means that we are looking for a function $f \in L_Y(K + A)$. Using Riemann-Roch we compute

$$\ell(K + A) = \ell(-A) + 2g_Y - 2 + \deg(A) - g_Y + 1 = g_Y - 1 + r + s,$$

where s is the number of wild ramified branch points. For such a selection of f we have that

$$\mathrm{div}(\pi^*(f\phi_1) = \pi^*\mathrm{div}(f\phi_1) + R \geq 0,$$

where R is the ramification divisor given by

$$R = \sum_{i=1}^{r} \sum_{P \mapsto b_i} \sum_{i=0}^{\infty} \big(e_i(P) - 1\big).$$

Moreover the divisor $\pi^*(f\phi_1)$ is G invariant, and we can select $f \in L_Y(K + A)$ such that is has polar divisor A. This imply that the support of $\pi^*(f\phi_1)$ has no intersection with the branch locus. □

We can now apply the method of S. Nakajima on $\Omega_X^{\otimes 2} = \Omega_X(\pi^*(f\phi_1))$. Let S be the set of points of the curve Y such that $\pi^{-1}(S) = \mathrm{supp}\big(\mathrm{div}(\pi^*(f\phi_1))\big)$. We follow the notation of [8]. Let $S_0 = \{b_1, \ldots, b_r\}$. For each $i = 1, \ldots, r$ we choose a point $P_i \in X$, satisfying $\pi(P_i) = b_i$, and let G_i be the decomposition group at P_i. We consider the $k[G]$-modules $k[G/G_i] = \{\sum_{\sigma \in G/G_i} a_\sigma \sigma\}$. We define surjective $k[G]$-homomorphisms $\Phi_i : k[G/G_i] \to k$, by $\Phi_i(\sum_{\sigma \in G/G_i} a_\sigma \sigma) = \sum_{\sigma \in G/G_i} a_\sigma$, and also $\Phi((\xi_1, \ldots, \xi_r)) = \sum_{i=1}^{r} \Phi_i(\xi_i)$.

Theorem 4.2. *Let G be a p-group. The Galois module structure of $\Omega_X^{\otimes 2}$ is determined by the following exact sequence:*

$$0 \to \Omega_X\big(-\pi^{-1}(S)\big) \to \Omega_X(-\pi^{-1}\big(S \cup S_0)\big) \to \ker\Phi \to 0, \tag{5}$$

where $\Omega_X(-\pi^{-1}\big(S \cup S_0)\big) \cong k[G]^{3g_Y - 3 + 2r}$.

Proof. Following the method of Nakajima, we define a $k[G]$-homomorphism

$$\phi : \Omega_X(-\pi^{-1}\big(S \cup S_0)\big) \to \bigoplus_{i=1}^{r} k[G/G_i],$$

by

$$\phi(\omega) = \bigoplus_{i=1}^{r} \left(\sum_{\sigma \in G/G_i} \mathrm{Res}_{\sigma P_i} \sigma \right), \quad \omega \in \Omega_X(-\pi^{-1}\big(S \cup S_0)\big).$$

Then $\ker(\phi) = \Omega_X\big(-\pi^{-1}(S)\big)$, and $\mathrm{Im}(\phi) = \ker\Phi$, using the residue and Riemann-Roch theorems.

For the $k[G]$-structure of $\Omega_X(-\pi^{-1}\big(S \cup S_0)\big)$ Theorem 1 of [8] implies that it is a free $k[G]$-module of rank $\gamma_Y - 1 + |S| + |S_0|$, where γ_Y is the p-rank of the Jacobian of Y. Since the curve is ordinary we have $g_Y = \gamma_Y$. On the other hand $|S| = 2g_Y - 2 + r$ and $|S_0| = r$. $\square$

The module in the middle of equation (5) is $k[G]$-projective, therefore it implies the following long exact sequence:

$$0 \to H_1(G, \ker\Phi) \to \left(\Omega_X^{\otimes 2}\right)_G \to k[G]_G^{3g_Y-3+2r} \to \ker\Phi_G \to 0.$$

This implies that the desired dimension can be computed:

$$\dim_k \left(\Omega_X^{\otimes 2}\right)_G = \dim_k H_1(G, \ker\Phi) + 3g_Y - 3 + 2r - \dim_k \ker\Phi_G. \tag{6}$$

We will use the sequence

$$0 \to \ker\Phi \to \bigoplus_{i=1}^{r} k[G/G_i] \xrightarrow{\Phi} k \to 0, \tag{7}$$

in order to compute the homology groups of $\ker\Phi$.

Equation (7), gives the long exact sequence:

$$H_2(G, \bigoplus_{i=1}^{r} k[G/G_i]) \xrightarrow{\psi_1} H_2(G,k) \to H_1(G, \ker\Phi) \to H_1(G, \bigoplus_{i=1}^{r} k[G/G_i]) \xrightarrow{\psi_2} \tag{8}$$

$$\xrightarrow{\psi_2} H_1(G,k) \to \ker\Phi_G \to \left(\bigoplus_{i=1}^{r} k[G/G_i]\right)_G \to k_G \to 1.$$

Using the above sequence we compute:

$$\dim_k H^1(G, \ker\Phi) = \dim_k \mathrm{Coker}\psi_1 + \dim_k \ker\psi_2.$$

It is known that $H_1(G,k) = \frac{G}{[G,G]} \otimes_{\mathbb{Z}} k$ and Hopf's theorem [12, 6.8.8] implies

$$H_2(G,k) = \frac{R \cap [F,F]}{[F,R]} \otimes_{\mathbb{Z}} k,$$

where $1 \to R \to F \to G \to 1$ is a free presentation of G.

Shapiro's lemma [2, p. 73], gives that $H_1(G, k[G/G_i]) = H_1(G_i, k)$. Moreover, since the curve in question is ordinary the ramification groups are elementary abelian, thus $H_1(G, k[G/G_i]) = G_i \otimes_{\mathbb{Z}} k$. Therefore, for the computation of the kernel of ψ_2 we have:

$$\psi_2 : \bigoplus_{i=1}^{r} G_i \otimes_{\mathbb{Z}} k \to \frac{G}{[G,G]} \otimes_{\mathbb{Z}} k.$$

The kernel of ψ_2 equals $\cap_{i=1}^{r}(G_i \cap [G,G]) \otimes_{\mathbb{Z}} k$. This is a group theoretic description of the kernel of ψ_2.

Using equation (8) we compute:

(9)

$$\dim_k H_1(G, \ker\Phi) = \dim_k \mathrm{coker}(\psi_1) + \sum_{i=r}^{r} \log_p |G_i| - \dim_k \frac{G}{[G,G]} \otimes_Z k + \dim_k \ker\Phi_G - r + 1.$$

Combining (6) with (9) we obtain

$$\dim_k(\Omega_X^{\otimes 2})_G = \dim_k \operatorname{coker}(\psi_1) + 3g_Y - 3 + r + \sum_{i=1}^{r} \log_p |G_i| - \dim_k \frac{G}{[G,G]} \otimes_{\mathbb{Z}} k + 1.$$

For the computation of the cokernel of ψ_1 we proceed as follows: For every ramification group fix a set of generators $F_i \in F$ and consider a set of relations R_i, such that $G_i = F_i/R_i$. Using Hopf's theorem for the computation of $H_2(G,k)$, the study of the map ψ_1 is reduced to the study of the map:

$$\bigoplus_{i=1}^{r} \frac{R_i \cap [F_i,F_i]}{[F_i,R_i]} \to \frac{R \cap [F,F]}{[F,R]}.$$

The groups G_i are elementary abelian. If G_i is a cyclic group of order p then it is immediate from Hopf's theorem that $H_2(G_i,k) = 0$.

We consider now the case of groups G_i that have at least two cyclic summands. Since the groups G_i are abelian, we have $[F_i,F_i] \subset R_i$, thus $R_i \cap [F_i,F_i] = [F_i,F_i]$. For a given $i = 1,\ldots,r$ consider the map

$$f_i : \frac{[F_i,F_i]}{[F_i,R_i]} \to \frac{R \cap [F,F]}{[F,R]}.$$

The kernel of f_i is $(\ker f_i = [F_i,F_i] \cap [F,R])/[F_i,R_i]$ and the image is isomorphic to

$$\mathrm{Im}(f_i) \cong \frac{[F_i,F_i]}{[F_i,F_i] \cap [F,R]} \otimes_{\mathbb{Z}} k \cong \frac{[F_i,F_i][F,R]}{[F,R]}$$

Combining this information together for all i such that $\log_p |G_i| > 1$ we obtain:

$$\operatorname{coker}\psi_1 = \frac{R \cap [F,F]}{\langle [F_i,F_i][F,R] \rangle} \otimes_{\mathbb{Z}} k.$$

where i runs over the ramification points such that $\log_p |G_i| \geq 1$. If all G_i have order p then the above formula reduces to Hopf's formula for $H_2(G)$. We collect all pieces of computation in the following

Proposition 4.3. *Let G be a p-group that is a subgroup of the automorphism group of an ordinary curve, and let G_i be the decomposition groups at the ramification points. Using the above notation we have for the dimension of covariant 2-differentials*

$$\dim_k \left(\Omega_X^{\otimes 2}\right)_G = 3g_Y - 3 + r + \sum_{i=1}^{r} \log_p |G_i| - \dim_k \frac{G}{[G,G]} \otimes_{\mathbb{Z}} k - 1 + \dim_k \frac{R \cap [F,F]}{\langle [F_i,F_i][F,R] \rangle} \otimes_{\mathbb{Z}} k.$$

Comparison of the computation done so far with the result of G.Cornelissen, F.Kato implies the following corollary:

Corollary 4.4. *Let G be a p-group that is a subgroup of the automorphism group of an ordinary curve, and let G_i be the decomposition groups at the ramification points. Suppose that $p > 3$. Using the above notation we obtain:*

$$\dim_k \frac{R \cap [F,F]}{\langle [F_i,F_i][F,R] \rangle} \otimes_{\mathbb{Z}} k = \dim_k \frac{G}{[G,G]} \otimes_{\mathbb{Z}} k - 1.$$

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